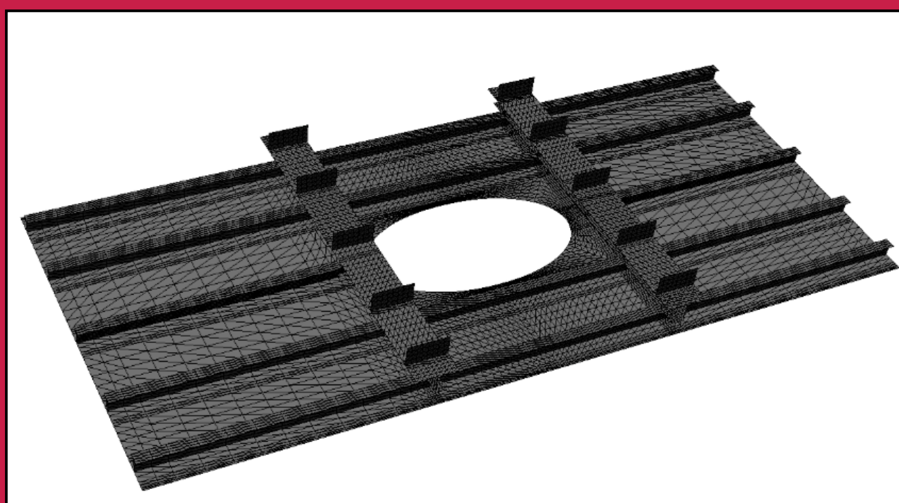
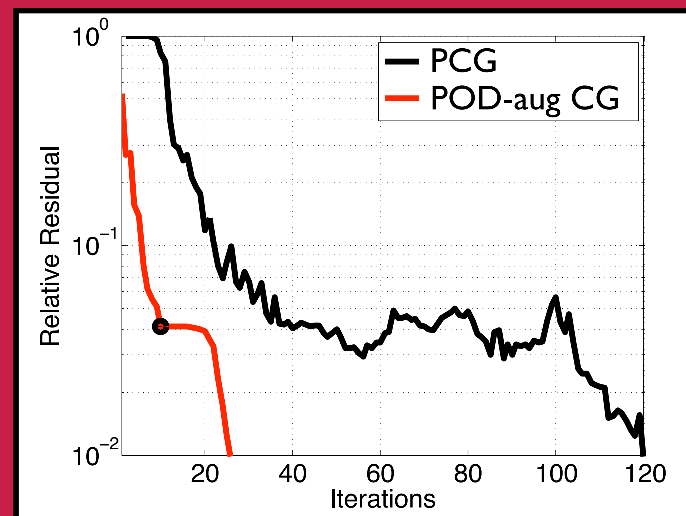
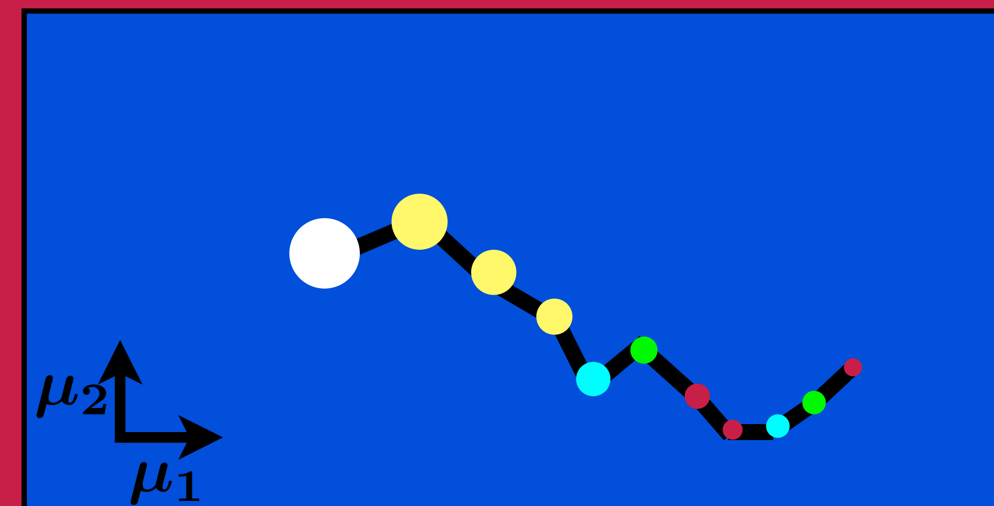
An adaptive POD-Krylov reduced-order modeling framework for repeated analyses problems







Kevin Carlberg and Charbel Farhat

Stanford University, USA

Department of Aeronautics & Astronautics

2009 Joint ASCE-ASME-SES Conference on Mechanics and Materials

Blacksburg, Virginia

June 27, 2009

Outline

- Real-time v. repeated analyses
  - Offline-online framework: ill-suited for repeated analyses
- Repeated analyses application: structural design optimization
- Novel adaptive POD-Krylov framework
  - POD-based iterative solver
  - Implementation
- Example: V-22 Osprey wing panel sensitivity analysis

Model reduction problem types

Real-time analysis

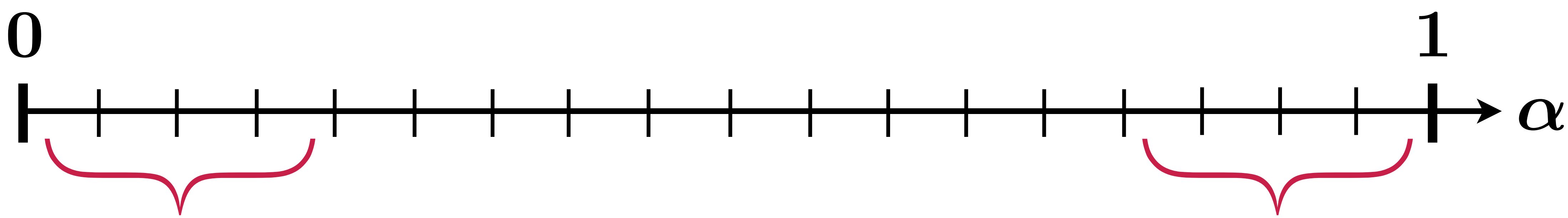
- "In-field" analysis
- Damage detection
- Model predictive control

Repeated analyses

- Design optimization
- Parameter space sampling
- Nonlinear analysis

Solution approach: competing objectives

$$\text{minimize} \quad \alpha \times \text{error} + (1 - \alpha) \times \text{cost}$$



Real-time analysis

$$\begin{aligned} \text{minimize} \quad & \text{error} \\ \text{s.t.} \quad & \text{online cost} \leq \tau \end{aligned}$$

Repeated analyses

$$\begin{aligned} \text{minimize} \quad & \text{total cost} \\ \text{s.t.} \quad & \text{error} \leq \epsilon \end{aligned}$$

Offline-Online framework

1) Offline
- Sample parameter space
- Build reduced basis

2) Online
- Analysis with reduced basis of fixed dimension

Real-time analysis

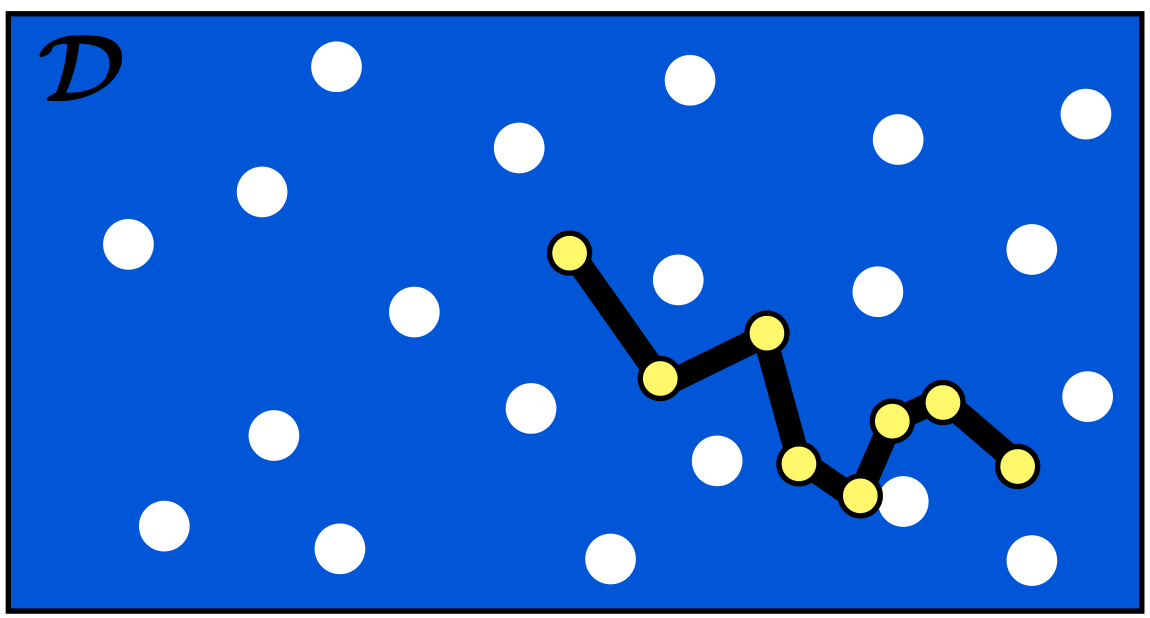
✓ High offline cost okay

✓ Very low online cost

Repeated analyses

✗ May preclude total cost savings

✗ May not meet accuracy requirements



$\mathcal{D}$ Parameter space

○ Full-order model evaluation

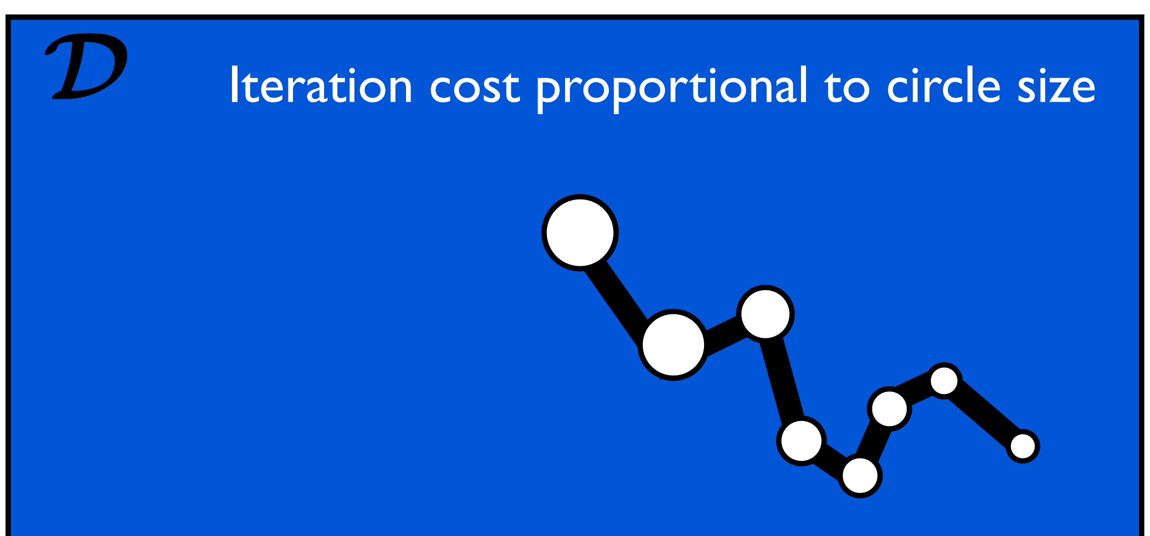
● Reduced-order model evaluation

— Online trajectory

→ Goal: total cost savings while satisfying accuracy requirements

Novel repeated analyses framework

- Guiding philosophy
  1. Avoid extra computations
  2. Fully exploit data generated from previous analyses
  3. Use data to accelerate convergence of analyses to specified accuracy
- Procedure
  1. Execute only required analyses (no offline sampling)
  2. Build a POD basis on-the-fly
  3. Use the POD basis within a novel POD-based iterative method to accelerate new analyses



$\mathcal{D}$ Parameter space

○ Accelerated analyses

— Trajectory

Application: structural optimization

- Structural optimization:

$$\underset{\mu \in \mathcal{D}}{\text{minimize}} \quad J\left(u(\mu), \mu\right)$$
$$\text{subject to} \quad l_i \leq c_i(u(\mu), \mu) \leq u_i, \quad i = 1, \ldots, n_c$$

- State equations $K(\mu)u = f(\mu)$ enforce dependence $u(\mu)$
  - $K(\mu)$ stiffness matrix
  - u state vector
  - $f(\mu)$ load vector
- Must repeatedly solve state and sensitivity equations

Gradient-based structural optimization

- At each optimization iteration k, solve

  1) State equations $K(\mu^{(k)})u = f(\mu^{(k)})$

  2) Sensitivity equations

     - Direct S.A. For $i = 1, \ldots, n_{\mathrm{vars}}$

$$K(\mu^{(k)})\frac{du}{d\mu_i} = \left.\frac{\partial f}{\partial \mu_i}\right|_{\mu^{(k)}} - \left.\frac{\partial K}{\partial \mu_i}\right|_{\mu^{(k)}} u$$

     or

     - Adjoint S.A. For $i = 1, \ldots, n_c + 1$

$$K(\mu^{(k)})\psi_i = \left.\frac{\partial \gamma_i}{\partial u}\right|_{\mu^{(k)}}^{T} \qquad \gamma_i = \begin{cases} c_i, & i = 1, \ldots, n_c \\ J, & i = n_c + 1 \end{cases}$$

Repeated analyses formulation

- For $k = 1, \ldots, K$ and $i = 1, \ldots, n_{\mathrm{RHS}}$, solve

$$\boldsymbol{K}(\boldsymbol{\mu}^{(k)})\boldsymbol{u}_i = \boldsymbol{f}_i(\boldsymbol{\mu}^{(k)})$$

  - $\boldsymbol{K}(\boldsymbol{\mu}^{(k)})$ large, sparse, symmetric positive definite (SPD)

- Iteratively solve by preconditioned conjugate gradient (PCG)
  - For $m = 1, \ldots, M$ (until convergence)

$$\underset{\boldsymbol{x} \in \mathcal{K}_m}{\mathrm{minimize}} \quad \frac{1}{2}\boldsymbol{x}^T \boldsymbol{K}(\boldsymbol{\mu}^{(k)})\boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{f}_i(\boldsymbol{\mu}^{(k)})$$

  - $\mathcal{K}_m$ Krylov subspace of dimension m
  - Final solution $\tilde{\boldsymbol{u}}_i \in \mathcal{K}_M$ satisfies specified solver tolerance

- Approach: accelerate PCG convergence using ROM concepts

POD-Krylov ROM Approach

Solve $K(\mu^{(k)})u_i = f_i(\mu^{(k)})$ for $k = 1, \ldots, K$, $i = 1, \ldots, n_{\mathrm{RHS}}$

- Compute approximations $\tilde{u}_i$ satisfying controlled tolerance ϵ_k

$$\frac{\|f_i(\mu^{(k)}) - K(\mu^{(k)})\tilde{u}_i\|_2}{\|f_i(\mu^{(k)}\|_2} < \epsilon_k$$

- Approximations lie in the sum of two subspaces

$$\tilde{u}_i \in \mathcal{P} + \mathcal{K}_M$$

  - $\mathcal{P}$ proper orthogonal decomposition (POD) subspace
  - "POD-Krylov reduced-order model"

- Compute $\tilde{u}_i$ very efficiently by a novel augmented conjugate gradient (CG) iterative method

Proper orthogonal decomposition

- Optimal representation of "snapshot" data
- Here, approximately minimize the projection error of the solution at a target configuration $\bar{\boldsymbol{\mu}}$ near $\boldsymbol{\mu}^{(k)}$

1. Snapshots $\{\boldsymbol{w}_j\}_{j=1}^{n_w}$: components of solution $\boldsymbol{u}(\bar{\boldsymbol{\mu}})$
   - Solutions from previous analyses
   - Sensitivity derivatives (Carlberg and Farhat, 2008)
2. Weights $\{\gamma_j\}_{j=1}^{n_w}$: estimate the solution

$$\boldsymbol{u}(\bar{\boldsymbol{\mu}}) \approx \boldsymbol{u}_{\text{est}}(\bar{\boldsymbol{\mu}}) = \sum_{j=1}^{n_w} \gamma_j \boldsymbol{w}_j$$

   - Radial basis functions & Taylor expansion coefficients
3. POD norm: $\|\boldsymbol{x}\|_{\boldsymbol{K}(\bar{\boldsymbol{\mu}})} \equiv \sqrt{\boldsymbol{x}^T \boldsymbol{K}(\bar{\boldsymbol{\mu}}) \boldsymbol{x}}$

POD bases and key properties

- Compute one POD basis for each RHS $i = 1, \ldots, n_{\mathrm{RHS}}$

$$\boldsymbol{\Phi}_i(n) \equiv \left[\boldsymbol{\phi}_1^i, \ldots, \boldsymbol{\phi}_n^i\right]$$

- Key properties

1. Optimal ordering
   - First n POD basis vectors span an optimal n-dimensional subspace

2. $\boldsymbol{K}(\bar{\mu})$-orthonormality

$$\boldsymbol{\Phi}_i(n)^T \boldsymbol{K}(\bar{\mu}) \boldsymbol{\Phi}_i(n) = \boldsymbol{I}$$

   - $\boldsymbol{\Phi}_i(n)^T \boldsymbol{K}(\mu) \boldsymbol{\Phi}_i(n) \approx \boldsymbol{I}$ for μ near $\bar{\mu}$

- Three stages to compute approximation $\tilde{\boldsymbol{u}}_i$ at $\boldsymbol{\mu}^{(k)}$ near $\bar{\boldsymbol{\mu}}$

1. Directly solve n_1-dimensional reduced equations (n_1 small)

$$\boldsymbol{\Phi}_i(n_1)^T \boldsymbol{K}(\mu^{(k)}) \boldsymbol{\Phi}_i(n_1) \hat{\boldsymbol{u}} = \boldsymbol{\Phi}_i(n_1)^T \boldsymbol{f}_i(\mu^{(k)}),$$

$$\tilde{\boldsymbol{u}}_{i,1} = \boldsymbol{\Phi}(n_1)\hat{\boldsymbol{u}}$$

- Accurate (Property 1) and low cost (n_1 small)

2. Iteratively solve n_2-dimensional reduced equations ($n_2 \gg n_1$)

$$\boxed{\boldsymbol{\Phi}_i(n_2)^T \boldsymbol{K}(\mu^{(k)}) \boldsymbol{\Phi}_i(n_2)} \hat{\boldsymbol{u}} = \boldsymbol{\Phi}_i(n_2)^T \left(\boldsymbol{f}_i(\mu^{(k)}) - \boldsymbol{K}(\mu^{(k)}) \tilde{\boldsymbol{u}}_{i,1} \right),$$

$$\tilde{\boldsymbol{u}}_{i,2} = \tilde{\boldsymbol{u}}_{i,1} + \boldsymbol{\Phi}_i(n_2)\hat{\boldsymbol{u}}$$

- Use augmented CG without forming reduced matrix
- More accurate (Property 1) and low cost (Property 2)

POD-augmented CG algorithm

3. Iteratively solve full state equations to specified tolerance ϵ_k

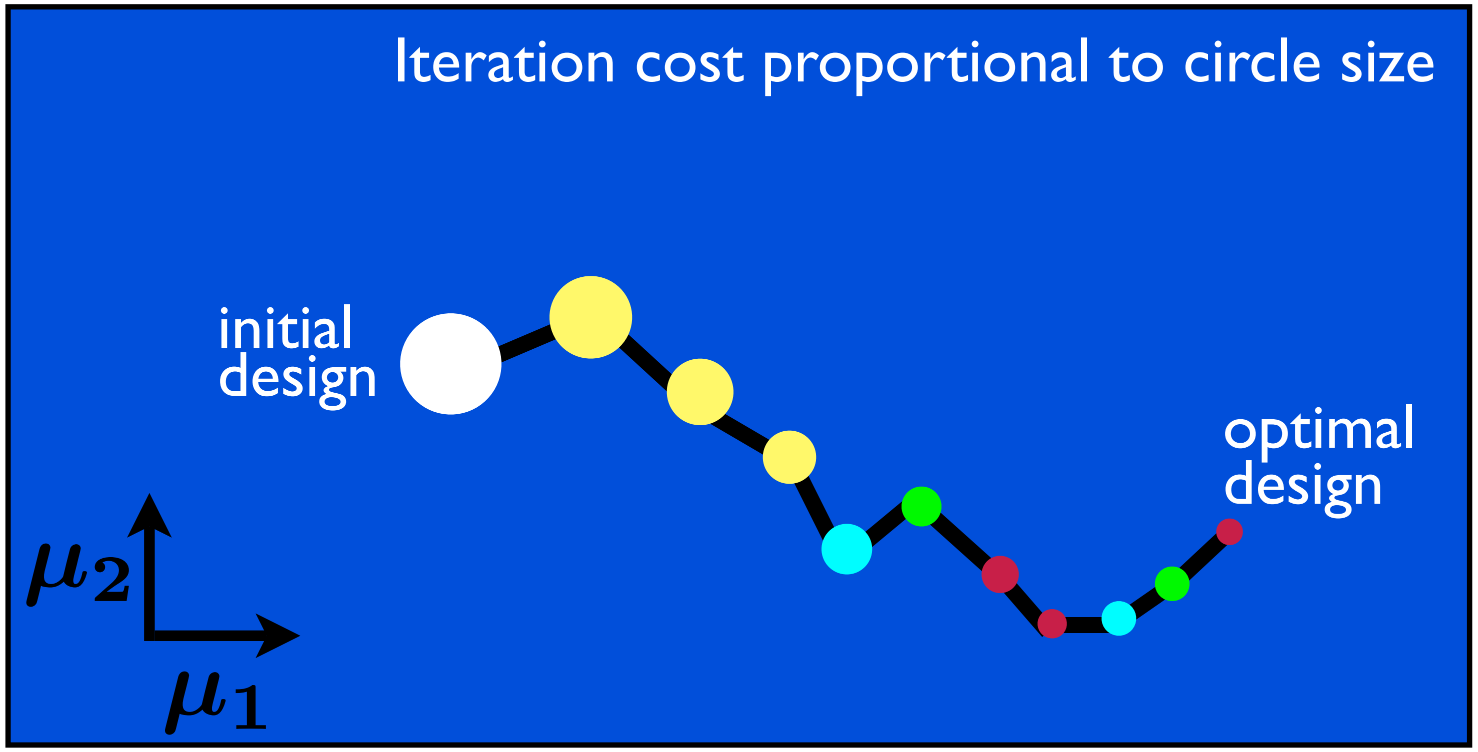
$$\boldsymbol{K}(\boldsymbol{\mu}^{(k)})\hat{\boldsymbol{u}} = \boldsymbol{f}_i(\boldsymbol{\mu}^{(k)}) - \boldsymbol{K}(\boldsymbol{\mu}^{(k)})\tilde{\boldsymbol{u}}_{i,2}$$
$$\tilde{\boldsymbol{u}}_i = \tilde{\boldsymbol{u}}_{i,2} + \hat{\boldsymbol{u}}$$

- Use augmented PCG (Farhat *et al.*, 1994)
- Provides "adaptivity" to meet any specified tolerance

- Multiple-RHS (solving state equations + sensitivity analysis)
  - Sequentially execute Stages 1-3 for $i = 1, \ldots, n_{\mathrm{RHS}}$
  - Stage 1 approximation space includes search directions from all previous RHS

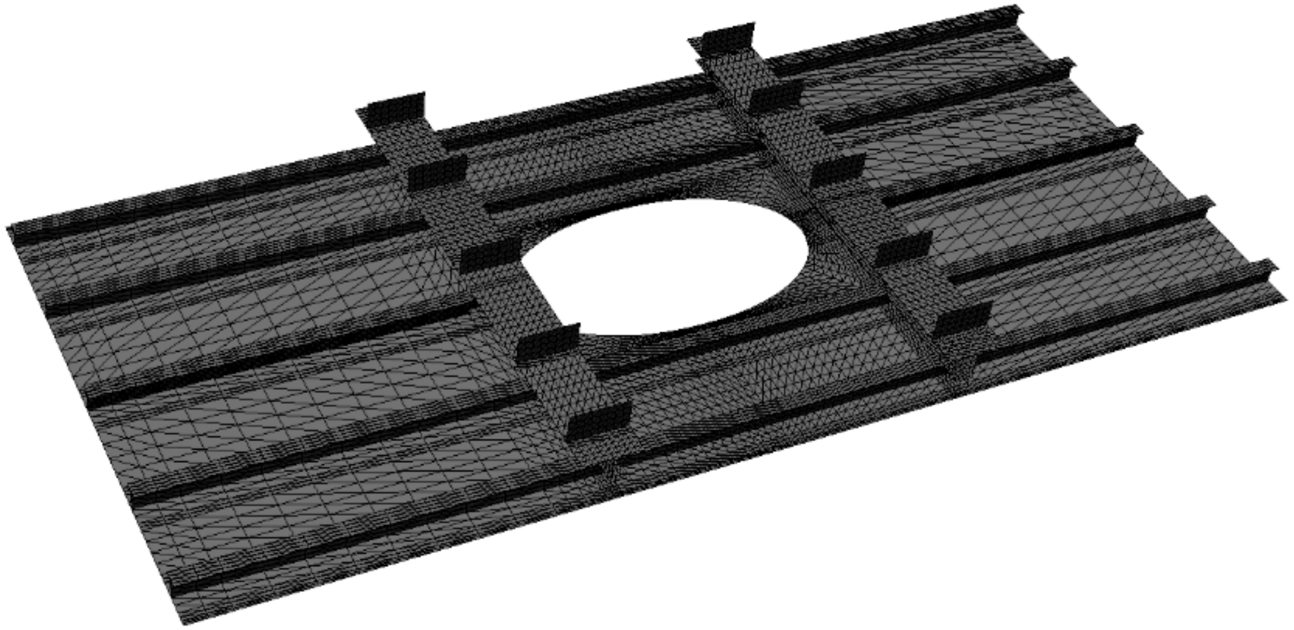
New framework in optimization

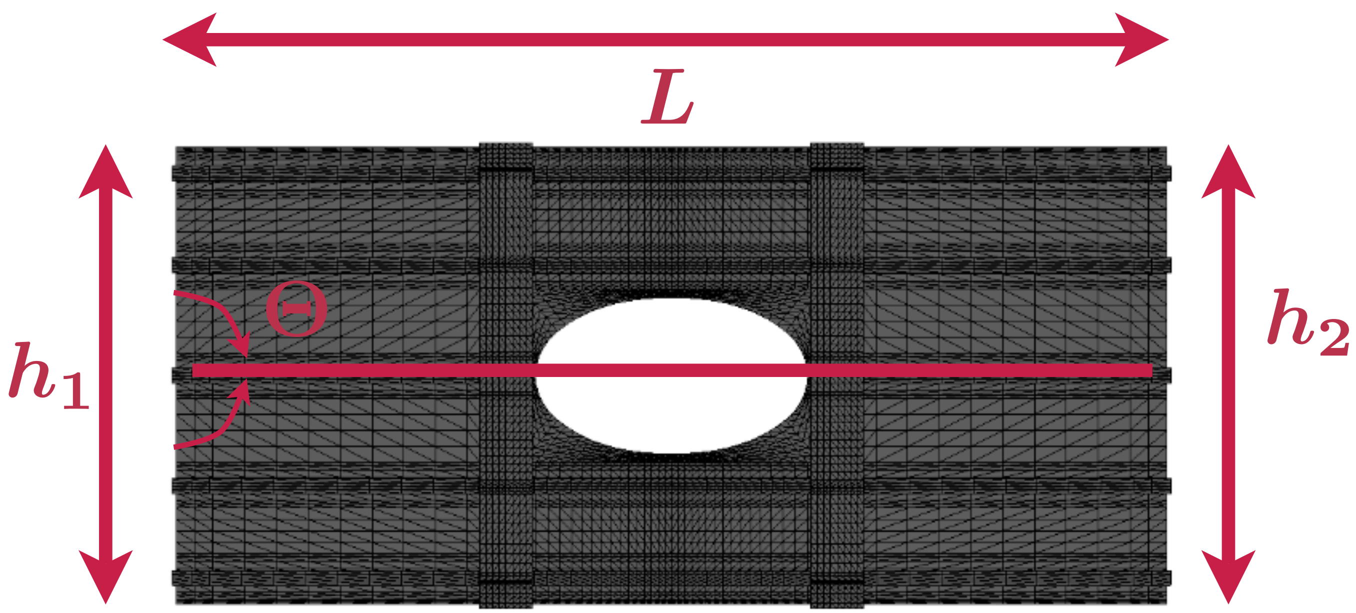
- Use relevant previous analyses to accelerate current analysis

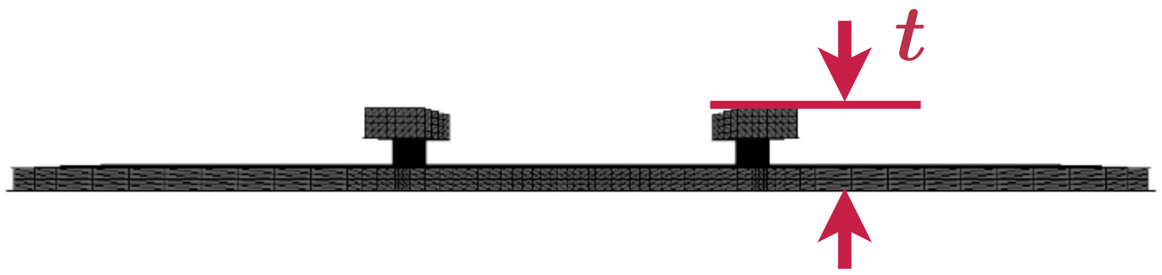


	Stage 1 basis	Stage 2 basis	Compute POD at end?
	W		
	W		✓
	$\Phi(n_1)$	$\Phi(n_2)$	
	$[W, \Phi(n_1)]$	$[W, \Phi(n_2)]$	

Example: V-22 Osprey wing panel



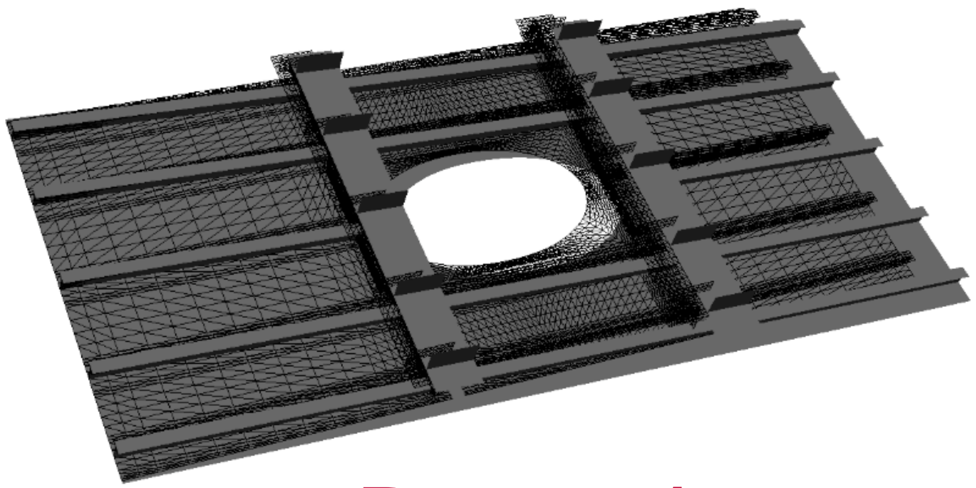




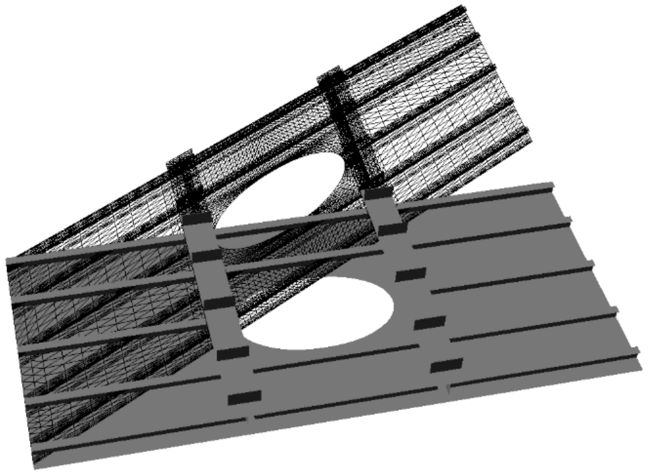
- Finite element model with 56,916 degrees of freedom
- 13 design variables (5 shape, 8 material)

Example: V-22 Osprey wing panel

- Problem Statement
  - Given: 10 previously-queried designs and 2 new designs



Design *A*



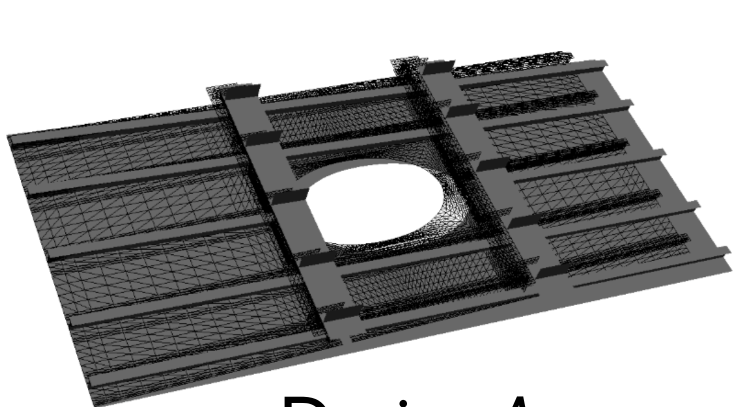
Design *B*

  - Compute: approximations $\tilde{u}_i, \; i = 1, \ldots, n_{\mathrm{RHS}}$ satisfying

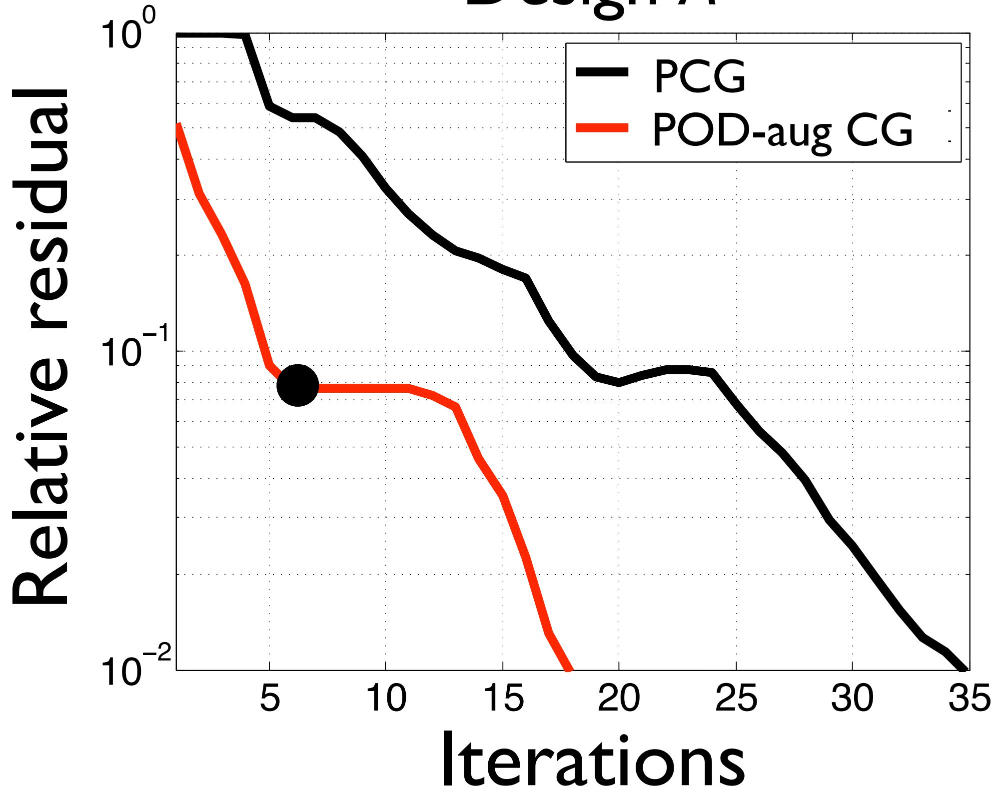
$$\frac{\|f_i(\mu) - K(\mu)\tilde{u}_i\|_2}{\|f_i(\mu)\|_2} < 10^{-2}$$

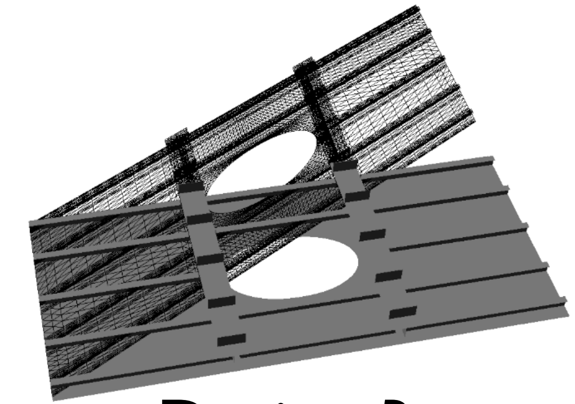
at the new designs

Results

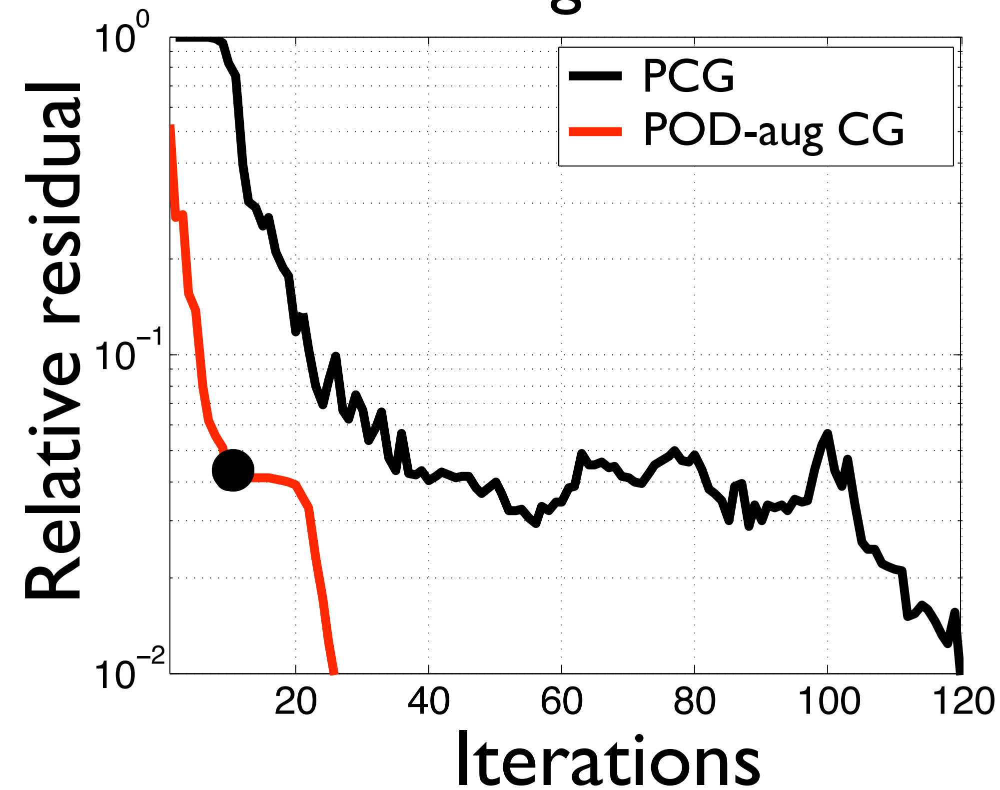


Design *A*





Design *B*



Error convergence $n_{\mathrm{RHS}} = 1$

- ● End of POD approximation

Simulation type	n_{RHS}	Speedup (flops), Design *A*	Speedup (flops), Design *B*
State equations	1	2.33	7.30
State equations + direct sensitivity	14	1.78	1.71

Conclusions

- A novel repeated analyses framework
  - Meets stated objectives:
    - ✓ Any accuracy requirement can be met
    - ✓ Guarantees cost savings
  - Efficiency due to choice of POD snapshots, weights, and norm
  - 1.7x to 7.3x speedup over existing iterative methods
- Future work
  - Fully implement for a repeated analyses problem
  - Combine with other augmented Krylov approaches (deflation)
  - Extend to systems with non-SPD matrices